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UNITED STATES DEPARTMENT OF THE INTERIOR BUREAU OF MINES HELIUM ACTIVITY HELIUM RESEARCH CENTER INTERNAL REPORT

NON-LINEAR REGRESSION AND THE PRINCIPLE OF LEAST SQUARES
THE METHOD OF EVALUATING THE CONSTANTS AND THE
CALCULATION OF VARIANCES AND COVARIANCES
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BRANCHFundamental Research
PROJECT NO. 4335

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Robert E. Barieau $\frac{1}{2}$ and B. J. Dalton $\frac{2}{2}$

ABSTRACT

This report is concerned with the necessary mathematical equations in order to accomplish the following objectives: 1. to evaluate the parameters such that the sum of the weighted squares of the residuals of an experimental observable be a true minimum, regardless of the functional relationship between the variables and these parameters;

2. to evaluate all variances and covariances of the parameters evaluated by means of the law for the propagation of errors; and 3. to insure that a true minimum will always be obtained. The equations presented in this report were developed for a three-parameter problem.

INTRODUCTION

The Helium Research Center, Bureau of Mines, has as one of its long-range objectives the development of an equation of state for

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Work on manuscript completed December 1965.

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helium that will reproduce the data to within the accuracy with which the data are known.

The Helium Research Center also has an experimental program for obtaining PVT data on gases and mixtures by the Burnett $(2)^{3/2}$ method.

3/ Underlined numbers in parentheses refer to items in the list of references at the end of this report.

In the Burnett method, one of the constants that must be evaluated is the volume-ratio of two containers. By the theory of the method, this constant is inherently non-linear. It was therefore decided to develop a capability for handling non-linear regression problems. This report gives the principles of the method the Helium Research Center uses in such problems.

Our method differs in several important respects from methods currently in use. In solving non-linear problems, the problem must be linearized and an iteration technique used to obtain the solution. All texts on non-linear regression, of which we are aware, linearize the problem before the normal equations are formed. This method is known as the Gauss-Newton method. In our method, the exact normal equations are formed, and the problem is linearized by expanding the exact normal equations in a Taylor's series expansion retaining the first two terms. This method is known as the Newton-Raphson method (7). The only work that we have been able to locate that uses this method in non-linear least squares problems is that of Strand, Kohl, and Bonham (8).

These two different methods, as will be shown later, lead to the same least squares solution provided the iteration procedure converges to an answer. We have found that if one starts within the region of convergence, the Newton-Raphson method converges more rapidly than the Gauss-Newton method. This is one advantage of the method we have chosen.

If on applying the Gauss-Newton or the Newton-Raphson method, the problem is diverging after the first iteration, a method must be found that will lead to convergence. One of the methods that may be tried at this stage is the negative gradient or method of steepest descent. If the step in the direction of the negative gradient is small enough, this method must lead to a smaller sum of the squares of the deviations. The problem with this method in the past has been deciding on the size of the step to be taken. If the Newton-Raphson method has been used in developing the normal equations for the first iteration, then the size of the step to be taken in the direction of the negative gradient can be evaluated very simply from the coefficients appearing in the normal equations. This is the second advantage of the method we have chosen.

The third advantage involves the calculation of the variances and covariances of the constants evaluated. As far as we are aware, all authors and all programs available calculate variances and covariances on the assumption that the formulas that apply to linear problems will apply to non-linear problems once the non-linear problems have been linearized. We reject this assumption, preferring to calculate

These two different methods, so will be shown later, lead on the one least squares converges some least squares solution provided the iteration procedure converges to an answer. We have found that it one starts within the region of convergence, the Newton Schman method converges were capably than the Causa-Newton method. This is one seventage of the method we have chosen.

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variances and covariances from the fundamental definition of these quantities and the law for the propagation of errors. (1, 5).

Some authors (6) claim that the least squares values of the constants evaluated in a non-linear problem are biased and should be corrected. All the proofs of this, that we have seen, assume the deviations are distributed with zero mean. This, of course, is never true in a non-linear problem unless this condition is imposed as a constraint. Further, the principle of least squares maximizes the probability that the deviations are equal to the true random errors. This is true for both linear and non-linear problems. This being true, we fail to see how any solution can be better than the least squares solution. We therefore take the least squares solution as being non-biased and apply no correction.

We have set the following objectives for our method.

- 1. To evaluate the parameters so that the sum of the weighted squares of the residuals of an experimental observable is a true minimum.
 - 2. Objective 1 is to be accomplished even though the functional relationship between the observables and parameters is such that the observable involved in the minimum of the sum of the weighted squares of the residuals cannot be explicitly expressed as a function of other observables and the parameters.
 - 3. All variances and covariances are to be calculated, with no approximations, by means of the law for the propagation of errors.
 - 4. The method is to be such that a true minimum will always be obtained.

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2. Objective I is to be occumplished even though the functional relationship between the observables and parameters is such that the observable involved in the minimum of the sum of the weighted equates of the residuals cannot be explicitly expressed as a function of other observables and the parameters.

A. The method is to be such that a true minimum will always to obtained.

This report is concerned with the necessary mathematical equations in order to accomplish the above-named objectives. The equations are developed for a three-parameter problem. The extension to the evaluation of more parameters should be obvious.

EVALUATION OF THE CONSTANTS

Suppose one has experimentally determined a set of n data points x_i , y_i . Let the functional relationship between x and y and the parameters A, B, and C be given by

$$F(y,x,A,B,C) = 0 (1)$$

We assume that there are no random errors in the x_i 's and that random errors occur in the observed y_i 's.

Now suppose we have evaluated the constants A, B, and C by some means or other. Then, because of random errors in y_i , equation (1) will not be exactly satisfied when the observed y_i and x_i values are substituted in equation (1). We will let F_i be the numerical value of F, when the observed y_i and x_i values are substituted in equation (1). Thus,

$$F_{i} = F(y_{i(0)}, x_{i}, A, B, C)$$
 (2)

Now when x_i is substituted in equation (1), we may solve for y_i so that equation (1) is satisfied exactly. We will designate this y_i as y_i (calc). Thus,

$$F(y_{i(calc)}, x_{i}, A, B, C) = 0$$
 (3)

The residual of y_i is given by

This report is concerned with the noteers or respective are equalized as a color to recomplish the above-cames objectives. The equations are developed for a chromosomerom problem. The extensive to the evelope evelope to the evelope.

STATULATION OF THE COLUMNS

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(1) 0 = (0, 0.4, 0.9)

New mappeds we make evaluated the tender of, H, and C in some means or other. Then, because of tender errors in p;, equation (1) will not be exactly satisfied when the constraint of our s; waters are substituted in equation (1) we will lat P; to the unscript of our s; when the charge of a special out of these the charge of a special out.

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Now when x, we nobstituted in equision (i), we may notice to a property of the will designate this verse as yelested exectly. We will designate this verse as yelested.

(3, 8, A, 1 (olean) () 3

$$Y_{i} = y_{i(0)} - y_{i(calc)}$$
 (4)

Now Y_i , the residual of y_i , is the difference between the observed and calculated values. This is not the true random error in our observed y_i because we do not know the true value of y_i . However, we can maximize the probability that our Y_i 's are equal to the true random errors, and this is just what the principle of least squares does. The principle of least squares says that we maximize the probability that the Y_i 's represent the true random errors by minimizing the sum of the weighted squares of the residuals.

Thus, the function to be minimized is given by

$$R = \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i}^{2}$$
 (5)

where w is the weight assigned to y i(o). R is a function of the constants to be evaluated: A, B, and C. The condition that R be a minimum is determined by

$$\frac{1}{2} \left(\frac{\partial R}{\partial A} \right)_{B,C} = \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial Y_{i}}{\partial A} \right) = 0$$
 (6)

$$\frac{1}{2} \left(\frac{\partial R}{\partial B} \right)_{A,C} = \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial Y_{i}}{\partial B} \right) = 0$$
 (7)

and

$$\frac{1}{2} \left(\frac{\partial R}{\partial C} \right)_{A,B} = \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial Y_{i}}{\partial C} \right) = 0$$
 (8)

New Y, the residual of y, is the difference between the observed and calculated values. This is not the true random error in our observed y because we do not know the true value of y. However, we can maximize the probability that our Y,'s are equal to the true random errors, and this is just what the principle of least squares says that we waithle the probability that the Y,'s represent the true random errors by miniprobability that the Y,'s represent the true random errors by miniprobability that the Y,'s represent the true random errors by miniprobability that the Wighted squares of the residuals.

Thus, the function to be utniced as given by

where w is the weight serigned to v (v). R is a function of the constants to be evaluated: A, B, and C. The condition that R be a minimum is determined by

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(a)
$$0 = \left(\frac{1}{2}\right)^{1/2} = \left(\frac{36}{50}\right)^{-1/2} = 0$$
 (a)

In the application of equations (6), (7), and (8), the observed y_i 's and x_i 's are to be held constant in the derivatives:

 $\left(\frac{\partial R}{\partial A}\right)_{B,C}$, $\left(\frac{\partial R}{\partial B}\right)_{A,C}$, $\left(\frac{\partial R}{\partial C}\right)_{A,B}$. Equations (6), (7), and (8) are the exact normal equations.

If Y_i is non-linear in the undetermined constants, then the solutions of our normal equations will not be straightforward. It will be necessary to solve them by an iterative technique in which values of the constants are assumed. This is done by expanding Y_i , $\left(\frac{\partial Y_i}{\partial A}\right)$, $\left(\frac{\partial Y_i}{\partial B}\right)$, and $\left(\frac{\partial Y_i}{\partial C}\right)$ in a Taylor's series expansion about an approximate solution, Y_i^O , retaining only the first two terms. Thus,

$$Y_{i} = Y_{i}^{o} + \left(\frac{\partial Y_{i}}{\partial A}\right)^{o} \Delta A + \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} \Delta B + \left(\frac{\partial C}{\partial Y_{i}}\right)^{o} \Delta C$$
 (9)

$$\left(\frac{\partial Y_{i}}{\partial A}\right) = \left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} + \left(\frac{\partial^{2} Y_{i}}{\partial A^{2}}\right)^{\circ} \Delta A + \left(\frac{\partial^{2} Y_{i}}{\partial A \partial B}\right)^{\circ} \Delta B + \left(\frac{\partial^{2} Y_{i}}{\partial A \partial C}\right)^{\circ} \Delta C \quad (10)$$

$$\left(\frac{\partial \mathbf{B}}{\partial \mathbf{B}}\right) = \left(\frac{\partial \mathbf{B}}{\partial \mathbf{B}}\right)^{\circ} + \left(\frac{\partial^{2} \mathbf{A}}{\partial \mathbf{B} \partial \mathbf{A}}\right)^{\circ} \Delta \mathbf{A} + \left(\frac{\partial^{2} \mathbf{A}}{\partial \mathbf{B}^{2}}\right)^{\circ} \Delta \mathbf{B} + \left(\frac{\partial^{2} \mathbf{A}}{\partial \mathbf{B} \partial \mathbf{C}}\right)^{\circ} \Delta \mathbf{C} \quad (11)$$

$$\left(\frac{\partial Y_{i}}{\partial C}\right) = \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + \left(\frac{\partial^{2} Y_{i}}{\partial C \partial A}\right)^{o} \Delta A + \left(\frac{\partial^{2} Y_{i}}{\partial C \partial B}\right)^{o} \Delta B + \left(\frac{\partial^{2} Y_{i}}{\partial C^{2}}\right)^{o} \Delta C \quad (12)$$

where the quantities $\triangle A$, $\triangle B$, and $\triangle C$ are defined as

$$\Delta A = A - A_{o}$$

$$\Delta B = B - B_{o}$$

$$\Delta C = C - C_{o}$$

In the application of equations (5), (7), and (8), the observed y,'s and x,'s are to be held constant in the derivatives:

(35) . (35) . (35) . (45) . Squattons (5), (7), and (8) are the

If V in non-linear in the understood constants, then the straightforward entoriors of our nerval equations will not be straightforward to solve them by an listactive reclambed in which values of the constants are assumed. This is done by use panding v, (34), (34), (35), (37), and (37) in a Taylor's sories expandation v, (34), (35), (37), and (37), and (37), and (37) in a Taylor's sories expandation of the constant and the const

$$(0) \qquad 20 \left(\frac{1}{100} \right) + 80 \left(\frac{1}{100} \right) + 20 \left(\frac{1}{100} \right) + 2$$

where the quantition (A, 45, and MC are delined as

where A, B, and C are our undetermined constants, and A $_{_{0}}$, B $_{_{0}}$, and C $_{_{0}}$ are approximate values for these quantities. Then to first order in the Δ 's

$$Y_{i}^{\circ} \left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} + \left[\left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial A^{2}}\right)^{\circ}\right] \Delta A$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial A}\right)^{\circ}\right] \Delta B$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial A}\right)^{\circ}\right] \Delta C$$

$$(13)$$

$$Y_{i} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} + \left[\left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial A \partial B}\right)^{\circ}\right] \Delta A$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial C}\right)^{\circ}\right] \Delta C$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + Y_{i}^{\circ} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial C}\right)^{\circ}\right] \Delta C$$

$$(14)$$

$$Y_{i}\left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + \left[\left(\frac{\partial Y_{i}}{\partial A}\right)^{\circ}\left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + Y_{i}^{\circ}\left(\frac{\partial^{2} Y_{i}}{\partial A\partial C}\right)^{\circ}\right] \Delta A$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + Y_{i}^{\circ}\left(\frac{\partial^{2} Y_{i}}{\partial C}\right)^{\circ}\right] \Delta B$$

$$+ \left[\left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + Y_{i}^{\circ}\left(\frac{\partial^{2} Y_{i}}{\partial C}\right)^{\circ}\right] \Delta C$$

$$(15)$$

where A, B, and C are our underermined constants, and A, E, and

O ere approximate values for these quantities. Then to first order

$$(21) \qquad \qquad (22) \qquad (32) \qquad (42) \qquad (42)$$

Substituting equations (13), (14), and (15) into our normal equations (6), (7), and (8), we have

$$a_1 \Delta A + b_1 \Delta B + c_1 \Delta C = m_1 \tag{16}$$

$$a_2 \Delta A + b_2 \Delta B + c_2 \Delta C = m_2 \tag{17}$$

$$a_3 \triangle A + b_3 \triangle B + c_3 \triangle C = m_3$$
 (18)

where

$$a_{1} = \sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial A} \right)^{o^{2}} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial A^{2}} \right)^{o} \right]$$
 (19)

$$a_{2} = b_{1} = \sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{o} \left(\frac{\partial Y_{i}}{\partial A} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial A \partial B} \right)^{o} \right]$$
 (20)

$$a_{3} = c_{1} = \sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial A} \right)^{o} \left(\frac{\partial Y_{i}}{\partial C} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial A \partial C} \right)^{o} \right]$$
(21)

$$b_2 = \sum_{i=1}^{n} w_{y_i(o)} \left[\left(\frac{\partial Y_i}{\partial B} \right)^{o^2} + Y_i^o \left(\frac{\partial^2 Y_i}{\partial B^2} \right)^{o} \right]$$
 (22)

$$b_{3} = c_{2} = \sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{o} \left(\frac{\partial Y_{i}}{\partial C} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial C} \right)^{o} \right]$$
 (23)

$$c_3 = \sum_{i=1}^{n} w_{y_i(o)} \left[\left(\frac{\partial Y_i}{\partial C} \right)^{o^2} + Y_i^o \left(\frac{\partial^2 Y_i}{\partial C^2} \right)^o \right]$$
 (24)

Substituting equations (13), (14), and (15) into our normal squations (6), (7), and (8), we have

produ

(25)
$$\left[\left(\frac{1}{2} \right)_{1}^{2} + \left(\frac{1}{2} \right)_{2}^{2} + \frac{1}{2} \left(\frac{1}{2} \right) \right] = 88$$

$$m_{1} = -\sum_{i=1}^{n} w_{y_{i}(o)} Y_{i}^{o} \left(\frac{\partial Y_{i}}{\partial A}\right)^{o}$$
 (25)

$$m_2 = -\sum_{i=1}^{n} w_{y_{i(o)}} Y_i^o \left(\frac{\partial Y_i}{\partial B}\right)^o$$
 (26)

$$m_3 = -\sum_{i=1}^{n} w_{y_{i(0)}} Y_i^{o} \left(\frac{\partial Y_i}{\partial C}\right)^{o}$$
 (27)

The solutions to equations (16), (17), and (18) are

$$D_{o}\Delta A = D_{1}^{m}_{1} + D_{2}^{m}_{2} + D_{3}^{m}_{3}$$
 (28)

$$D_{o}^{\Delta B} = D_{4}^{m}_{1} + D_{5}^{m}_{2} + D_{6}^{m}_{3}$$
 (29)

$$D_{o}\Delta C = D_{7}^{m_{1}} + D_{8}^{m_{2}} + D_{9}^{m_{3}}$$
 (30)

where

$$D_1 = b_2 c_3 - b_3 c_2 \tag{31}$$

$$D_2 = b_3 c_1 - b_1 c_3 \tag{32}$$

$$D_3 = b_1 c_2 - b_2 c_1 \tag{33}$$

$$D_4 = a_3 c_2 - a_2 c_3 (34)$$

$$D_5 = a_1 c_3 - a_3 c_1 \tag{35}$$

$$D_6 = a_2 c_1 - a_1 c_2 (36)$$

$$D_7 = a_2b_3 - a_3b_2 \tag{37}$$

$$D_8 = a_3 b_1 - a_1 b_3 \tag{38}$$

$$D_9 = a_1 b_2 - a_2 b_1 \tag{39}$$

(12)
$$(\frac{2}{3})^{\frac{1}{2}}$$
 (0) (2) (2)

The solutions to equations (16), (17), and (18) are

$$D_{ab} = D_{c} m_{c} + D_{c} m_{c} + D_{c} m_{d} = A A_{c} G$$
 (28)

$$(es) \qquad \qquad \epsilon^{m_1}G + \epsilon^{m_2}G + \epsilon^{m_3}G = B\Delta_{\alpha}G$$

$$(30) \qquad \qquad (30) + p_{g_{1}} + p_{g_{2}} + p_{g_{3}} = 31.0$$

and resident

$$(20)$$
 $e^{d_2n} - e^{d_2n} = e^0$

$$a_{20} = a_{10} - a_{20} = a_{20}$$
 (39)

$$D_{o} = a_{1}D_{1} + a_{2}D_{2} + a_{3}D_{3}$$
 (40)

The solutions of equations (28), (29), and (30) give the corrections to be applied to the assumed values of our undetermined constants.

In the Gauss-Newton method of linearization, the second term in the summations of the a's, b's, and c's of equations (19)-(24) is neglected; this does not lead to an error as long as the method converges because the exact solutions of equations (16), (17), and (18) are: $\triangle A = \triangle B = \triangle C = 0$.

When the functional relationship between the observables is such that $y_{i(calc)}$ cannot be solved for explicitly, it will be necessary to solve equation (3) for $y_{i(calc)}$ by a series of approximations. Let us expand F in a truncated Taylor's series expansion about the point x_i , $y_{i(o)}$:

$$F = F_{i} + \left(\frac{\partial F}{\partial y}\right)_{\substack{x,A,B,C \\ y=y_{i}(o) \\ x=x_{i}}} \left(y_{i(calc)} - y_{i(o)}\right) = 0$$
 (41)

or

$$y_{i(o)} - y_{i(calc)} = \frac{F_{i}}{\left(\frac{\partial F}{\partial y}\right)_{x,A,B,C}}$$

where in equations (41) and (42) the symbol
$$\left(\frac{\partial F}{\partial y}\right)_{\substack{x,A,B,C\\y=y_i(o)\\x=x_i}}$$
 means that

The solutions of equations (28), (39), and (30) give the corrections to be applied to the sesumed values of our modetermined constants.

In the summerium of the a's, b's, and c's of equations (19)-(24) is neglected; this does not lead to an error as long as the method converges because the exact solutions of equations (16), (17), and (18) are: AA - AA - AA - AB - AC - D.

When the functional relationship between the observables is such
that fi(calc) cannot be solved for explicitly, it will be necessary
to solve equation (3) for y₁(calc) by a series of approximations.

Let us expand I in a truncated Taylor's series expansion about the
point X₁, Y₁(s):

(26) = (26) = (27) (27) (27) (27) (27) (27)

Jail street (35) lodays of (54) but (14) smillsupe at each

the derivative of F with regard to y, keeping x, A, B, and C constant, is to be evaluated at the point

$$y = y_{i(0)}$$

$$x = x_{i}$$

The solution of equation (42) gives the first approximation for $y_{i(calc)}$. We designate this value as $y_{i(calc)_1}$. This value is then substituted into equation (3), and if $y_{i(calc)_1}$ is not the exact answer, equation (3) will not be satisfied exactly. We designate this value of F as $F_{i(calc)_1}$. Thus,

$$F_{i(calc)_1} = F(x_i, y_{i(calc)_1}, A, B, C)$$
 (43)

Then the second approximation, $y_{i(calc)_{2}}$, of $y_{i(calc)}$ is obtained from the expression

$$y_{i(calc)_{1}} - y_{i(calc)_{2}} = \frac{F_{i(calc)_{1}}}{\left(\frac{\partial F}{\partial y}\right)_{\substack{x,A,B,C\\y=y_{i(calc)_{1}}\\x=x_{i}}}}$$
(44)

This iteration is repeated until equation (3) is satisfied to within any amount we wish to specify.

Once we have $y_{i(calc)}$ and $y_{i(o)}$, then $Y_{i} = y_{i(o)} - y_{i(calc)}$, and Y_{i}^{2} can be calculated. Then if $w_{i(o)}$ is known or has been assigned, $y_{i(o)}$ R may be calculated by means of equation (4).

the derivative of F with regard to y, keeping w. A. D. and C constant.

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The solution of equation (42) gives the tiret approximation for Yicaic). We designate this value as vicaicly. This value is then substituted into equation (3), and if yicaic), is not the exact answer, equation (3) will not be satisfied exactly. We designate this value of F as Ficaic), Thus,

(Colo), " P(N, N, (colo)), A, B, C)

Then the second approximation, yi(calc), of yi(calc) is obtained from the expression

(40) (20) Talesta Tale

This teather is reginted notif equation (3) is extining an within any amount we wish to epocify.

Once we have Pi(calc) and Pi(c): then Y = Vi(calc) - Pi(calc) and C can be calculated. Then if we have been assigned, I may be calculated by scene of equation (A).

If the $y_{i(0)}$'s all have the same precision index, they will all have the same weight and $w_{i(0)} = 1$. If the $y_{i(0)}$'s do not all have the same precision index, then

$$w_{y_{i(o)}} = \frac{L^2}{S_{y_{i(o)}}^2}$$
(45)

where L is a constant and $S_{i(0)}^2$ is the variance of $y_{i(0)}$. In a particular problem, it may be necessary to assume that $w_{i(0)} = 1$ in the beginning. However, if this is done, the residuals, $\left[y_{i(0)} - y_{i(calc)}\right]$, should be examined to see if there is any statistical evidence for the residuals squared being a function of y. Any assumption as to the variance, $S_{y_{i(0)}}^2$, being a function of y can always be checked by examining the residuals. In any event, $w_{i(0)}$ is not a function of the constants to be evaluated.

We now proceed to develop the equations needed to calculate the coefficients of our normal equations. Differentiating equation (4) with regard to A, keeping x_i , $y_{i(o)}$, B, and C constant, we have

$$\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i},y_{i}(o),B,C} = -\left(\frac{\partial y_{i}(calc)}{\partial A}\right)_{x_{i},B,C} \tag{46}$$

Differentiating equation (3) with regard to A, keeping \mathbf{x}_{i} , B, and C constant, we have

$$\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C} \left(\frac{\partial y_{i(calc)}}{\partial A}\right)_{x_{i},B,C} + \left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i(calc)},B,C} = 0 \quad (47)$$

If the y₁(o)'s all have the same precision index, they will all have the same weight and w = 1. If the y₁(o)'s do not all have the same precision index, then

where L is a constant and 3 (0) is the variance of y₁(0). In a particular problem, it may be necessary to assume that w = 1 in the beginning. However, if this is done, the residuals, y₁(0) (y₁(0) - y₁(caic)), should be examined to see if there is any arotistical evidence for the residuals squared oning a function of y. Any assumption as to the variance, 8 , being a function of y can always be checked by examining the residuals. In any event, w always be checked by examining the residuals. In any event, w (10) and a function of the constants to be evaluated.

We now proceed to develop the squattons needed to calculate the coefficients of our normal equations. Differentiating equation (4) with regard to A, beeping we, yeld, B, and C constant, we have

Differentiating equation (3) with regard to A. heeping x. B. and C constant, we have

or

$$\left(\frac{\partial y_{i(calc)}}{\partial A}\right)_{x_{i},B,C} = -\frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i(calc)},B,C}}{\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C}}$$
(48)

and if we substitute equation (48) in equation (46), we have

$$\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i},y_{i(o)},B,C} = \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i(calc)},B,C}}{\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C}}$$
(49)

Similarly, it can be shown that

$$\left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{i},y_{i}(o),A,C} = \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc),A,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}$$
(50)

and that

$$\left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i},y_{i}(o),A,B} = \frac{\left(\frac{\partial F}{\partial C}\right)_{x_{i},y_{i}(calc),A,B}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}$$
(51)

Differentiating equation (49) with regard to A, keeping x_i , $y_i(0)$, B, and C constant, we have.

and if we substitute aquation (68) in equation (46), we have

Similarly, it can be shown that

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Differentiating equation (A9) with regard to A, hosping with 'i(o)' B, and C constant, we have.

$$\frac{\left(\frac{\partial^{2}Y_{i}}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} = \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C}{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C} = \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C}{\left(\frac{\partial F}{\partial A}\right)_{x_{i},A,B,C},A,B,C} \times_{i},B,C}$$

$$\frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} \times_{i},A,B,C} \times_{i},B,C} \times_{i},B,C} \times_{i},B,C} \times_{i},B,C} \times_{i},A,B,C} \times_{$$

Now

$$\left(\frac{\partial F}{\partial A}\right)_{x_i,y_i(calc),B,C} = G(y_i(calc),A,B,C)$$

$$\left[d\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C,x_{i},B,C}\right]^{dA}_{x_{i},y_{i}(calc)} = \begin{bmatrix} \frac{\partial}{\partial A}\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C,x_{i},y_{i}(calc)} \\ + \begin{bmatrix} \frac{\partial}{\partial y_{i}(calc)}\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C,x_{i},A,B,C} \end{bmatrix}^{dy_{i}(calc)}_{dy_{i}(calc)}$$
(53)

and

$$\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C_{x_{i},B,C} = \left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C_{x_{i},y_{i}(calc)},B,C_{x_{i},y_{i}(calc)},B,C_{x_{i},A,B,C}\right)_{x_{i},B,C}$$

$$\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C_{x_{i},A,B,C} \left(\frac{\partial y_{i}(calc)}{\partial A}\right)_{x_{i},B,C}$$

Also,

$$\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C} = g(y_{i(calc)},A,B,C)$$

$$\left[d\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{dy_{i}(calc)}\right]_{x_{i},A,B,C}^{dy_{i}(calc)} = \begin{bmatrix} \frac{\partial}{\partial y_{i}(calc)} & \frac{\partial F}{\partial y_{i}(calc)} \\ + \left[\frac{\partial}{\partial A} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{dy_{i}(calc)} \right]_{x_{i},A,B,C}^{dy_{i}(calc)} \\ + \left[\frac{\partial}{\partial A} \left(\frac{\partial F}{\partial A}\right)_{x_{i},A,B,C}^{dy_{i}(calc)} \right]_{x_{i},A,B,C}^{dy_{i}(calc)} \\ + \left[\frac{\partial}{\partial A} \left$$

and

$$\frac{\left[\frac{\partial}{\partial A}\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}\right]_{x_{i},A,B,C}}{\left[\frac{\partial}{\partial A}\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}\right]_{x_{i},A,B,C}} \times_{i},y_{i}(calc),B,C} + \frac{\partial}{\partial y_{i}(calc)}\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} \times_{i},A,B,C} \times_{i},A,B,C}$$

When we substitute equations (54) and (56) into equation (52), we find

Also.

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When we aspected advertions (29) and (49) the dataseton (25) . An

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$$\frac{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}\left(\frac{\partial^{2}F}{\partial A^{2}}\right)_{x_{i},Y_{i}(calc)}^{x_{i},B,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}} + \frac{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}\left(\frac{\partial^{2}F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}\left(\frac{\partial Y_{i}(calc)}{\partial A}\right)_{x_{i},B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},Y_{i}(calc)}^{x_{i},A,B,C}\left(\frac{\partial^{2}F}{\partial A^{2}Y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},Y_{i}(calc)}^{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},Y_{i}(calc)}^{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}} + \frac{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)$$

But from equation (48)

$$\left(\frac{\partial y_{i(calc)}}{\partial A}\right)_{x_{i},B,C} = -\frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i(calc)},B,C}}{\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C}}$$
(48)

But from equation (44)

When we substitute equation (48) in equation (57), we see that

$$\frac{\left(\frac{\partial^{2} F}{\partial A^{2}}\right)_{x_{i},y_{i}(calc)},B,C}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} = \frac{2\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C} \left(\frac{\partial^{2} F}{\partial A \partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} = \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)},B,C} = \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial A}\right)_{x_{i},A,B,C}} = \frac{2\left(\frac{\partial F}{$$

Similarly,

$$\frac{\left(\frac{\partial^{2} F}{\partial B^{2}}\right)_{x_{i},y_{i}(calc)},A,C}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} - \frac{2\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)},A,C} \left(\frac{\partial^{2} F}{\partial B\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2} + \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)},A,C}^{2} \left(\frac{\partial^{2} F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)}^{2} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2}} + \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)}^{2} \left(\frac{\partial^{2} F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2}} + \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)}^{2} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{2}} + \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B,C}^{2}}{\left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B,C}^{2}} + \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B$$

and

When we substitute equation (48) in equation (57); we see that

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$$\begin{vmatrix}
\frac{\partial^{2}F}{\partial c^{2}} \\
\frac{\partial^$$

Differentiating equation (49) with regard to B, keeping $x_i, y_i(0), A, C$ fixed, we have

$$\begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ x_{i}, y_{i}(calc) \\ x_{i}, x_{i}, y_{i}(calc) \\ x_{i}, x_{i}$$

Let us differentiate $(\frac{\partial F}{\partial A})$ with regard to B, keeping x_i, x_i, y_i(calc), B,C A, and C constant. When we do this, we get

$$\begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ x_{i}, y_{i}(calc)
\end{bmatrix}_{x_{i}, y_{i}(calc)} = \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ x_{i}, y_{i}(calc)
\end{bmatrix}_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, x_{i}, y_{i}(calc)} + \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial A}\right)_{x_{i}, y_{i}(calc)}, B, C \\ \frac{\partial}{\partial A}\right)_{x_{i}, x_{i}, x_{i},$$

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Differentiating suinties (as) with regard on A, louping within 1910) with the base of the base

Let us differentiate (EE) surprise out to B. bespins a. .

Constitution (Consti

But

$$\left(\frac{\partial y_{i(calc)}}{\partial B}\right)_{x_{i},A,C} = -\frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i(calc)},A,C}}{\left(\frac{\partial F}{\partial y_{i(calc)}}\right)_{x_{i},A,B,C}}$$
(63)

and if we substitute equation (63) in equation (62), we see that

$$\begin{bmatrix}
\frac{\partial}{\partial B} \begin{pmatrix} \frac{\partial F}{\partial A} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, B, C, x_{i}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial B} \end{pmatrix}_{x_{i}, y_{i} \text{ (calc)}}, A, C \\
- \begin{pmatrix} \frac{\partial^{2} F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}} \end{pmatrix}_{x_{i}, A, B, C} \begin{pmatrix} \frac{\partial F}{\partial A \partial y_{i} \text{ (calc)}}$$

Now if we differentiate $\left(\frac{\partial F}{\partial y_{i(calc)}}\right)$ with regard to B, holding x_{i} , A, and C constant, we get

$$\begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial y_{i}(calc)} \right)_{x_{i},A,B,C} \\
x_{i},A,B,C \end{bmatrix}_{x_{i},A,B,C} \\
+ \begin{bmatrix}
\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial y_{i}(calc)} \right)_{x_{i},A,B,C} \\
\frac{\partial}{\partial y_{i}(calc)} \left(\frac{\partial F}{\partial y_{i}(calc)} \right)_{x_{i},A,B,C} \\
x_{i},A,B,C \end{bmatrix}_{x_{i},A,B,C} \\
\begin{pmatrix}
\frac{\partial y_{i}(calc)}{\partial B} \\
\frac{\partial y_{i}(calc)}{\partial B} \\
x_{i},A,C
\end{pmatrix}$$
(65)

or, substituting equation (63) in equation (65),

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and if we substitute equation (63) in constion (62), we see that

now if we differentiate (The color) with regard on a to woll with the color of the

or, substituting aquation (63) to equation (65),

$$\left[\frac{\partial}{\partial B}\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}^{x_{i},A,C}\right] = \left(\frac{\partial^{2} F}{\partial B \partial y_{i}(calc)}\right)_{x_{i},A,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)}^{x_{i},A,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)}^{x_{i},A,B,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B,C}^{x_{i},A,B,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B,C}^{x_{i},A,B,C}^{x_{i},A,B,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},A,B,C}^{x_{i},A,B,C} \left(\frac{\partial F}{\partial B}\right$$

Therefore, if we substitute equations (64) and (66) in equation (61), equation (61) is expressible as

$$\frac{\left(\frac{\partial^{2}F}{\partial A\partial B}\right)_{x_{i},y_{i}(calc),C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc),A,C} \left(\frac{\partial^{2}F}{\partial A\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial A\partial B}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc),B,C} \left(\frac{\partial F}{\partial B\partial y_{i}(calc)}\right)_{x_{i},A,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} + \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc),B,C} \left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc),A,C} \left(\frac{\partial^{2}F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}$$

$$(67)$$

Therefore, if we substitute equations (64) and (66) in equation (61), equation (61) is expressible as

We can derive expressions for $\left(\frac{\partial^2 Y_i}{\partial A \partial C}\right)_{x_i,y_i(o)}^{and}$ and $\left(\frac{\partial^2 Y_i}{\partial B \partial C}\right)_{x_i,y_i(o)}^{and}$ similar to equation (67). The results are

$$\left(\frac{\partial^{2} F}{\partial A \partial C}\right)_{x_{i},y_{i}(calc)} = \frac{\left(\frac{\partial F}{\partial C}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial^{2} F}{\partial C}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial^{2} F}{\partial A \partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial A \partial y_{i}(calc)}\right)_{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial^{2} F}{\partial C \partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}} - \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial^{2} F}{\partial C \partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}} + \frac{\left(\frac{\partial F}{\partial A}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial F}{\partial C}\right)_{x_{i},y_{i}(calc)} \left(\frac{\partial^{2} F}{\partial C \partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial Y_{i}(calc)}\right)_{x_{i},A,B,C}}$$
(68)

$$\frac{\left(\frac{\partial^{2}F}{\partial B\partial C}\right)_{x_{i},y_{i}(calc),A}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} \frac{\left(\frac{\partial^{2}F}{\partial B\partial y_{i}(calc)}\right)_{x_{i},A,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}} \frac{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)},A,C} \frac{\left(\frac{\partial^{2}F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}$$

$$+ \frac{\left(\frac{\partial F}{\partial B}\right)_{x_{i},y_{i}(calc)},A,C}{\left(\frac{\partial F}{\partial C}\right)_{x_{i},y_{i}(calc)},A,B} \frac{\left(\frac{\partial^{2}F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}{\left(\frac{\partial F}{\partial y_{i}(calc)}\right)_{x_{i},A,B,C}}$$

$$(69)$$

The method of obtaining a least squares solution for the constants A, B, and C is as follows.

We have n pairs of the observed quantities, x and y i(o).

- 1. We assume values for A, B, and C.
- 2. We then calculate the n values of F_i , using equation (2).
- 3. We next calculate the n values of $y_{i(calc)}$ using an iterative method involving equations (42), (43), and (44).
 - 4. We next calculate the n values of Y, from equation (4).
 - 5. We next calculate the n values of Y_i^2 .
- 6. Using designated values for w , we next calculate R, $y_{i(o)}$ the sum of the weighted squares of the residuals, using equation (5).
- 7. We next calculate the n values of $(\partial F/\partial y_i)$ by differentiation of the analytical expression for F, keeping x_i , A, B, and C constant.
- 8. We next calculate the n values of $(\partial F/\partial A)$, evaluated at $y_{i(calc)}$, by differentiation of the analytical expression for F, keeping x_i , $y_{i(calc)}$, B, and C constant.
- 9. We next calculate the n values of $(\partial F/\partial B)$, evaluated at $y_{i(calc)}$, by differentiating the analytical expression for F, keeping x_i , $y_{i(calc)}$, A, and C constant.
- 10. We next calculate the n values of $(\partial F/\partial C)$, evaluated at $y_{i(calc)}$, by differentiation of the analytical expression for F, keeping x_i , $y_{i(calc)}$, A, and B constant.
- 11. We next calculate the n values of $(\partial^2 F/\partial A^2)$, evaluated at $y_{i(calc)}$, by differentiating the analytical expression for F twice.

The method of obtaining a least squares solution for the con-

We have a pairs of the observed quantities, x and y (a)

- 1. We assume velues for A, H, and C
- 2. We then calculate the n values of F, deing equation (2).
- 3. We next calculate the a values of y (calc) using an iterawe marked involving aquations (42), (43), and (44).
 - A. We next calculate the n values of Y, from squatfon (A)
 - 5. We mant culculate the n values of Y.
- the sum of the weighted aquares of the residuals, using equation (5).
 - 7. We next calculate the n values of (6F/59x(calc)) by differentiation of the analytical expression for F, keeping x, A. B. and C constant.
 - 8. We now t calculate the n values of (3F/3A), evaluated at y(calc), by differentiation of the analytical expression for F, keeping x1, y(calc), 8, and C constant.
 - 9. We next colculate the n values of (35/35), evaluated of 71(celc) thy differentiating the analytical expression for E. heaping x, Nicelc) A, and C constant.
 - 10. We next coloulate the n values of (6F/5C), evaluated at vicale), by differentiation of the enalytical expression for F, keeping x, y greater, A, and B constant.
 - II. We next calculate the a values of (6°F/6A°), evaluated at Ticolo), by differentiating the analytical expression for F twice.

- 12. We next calculate the n values of $(\partial^2 F/\partial B^2)$, evaluated at $y_{i(calc)}$, by differentiation of the analytical expression for F twice.
- 13. We next calculate the n values of $(\partial^2 F/\partial C^2)$, evaluated at $y_{i(calc)}$, by differentiating the analytical expression for F twice.
- 14. We next calculate the n values of $(\partial^2 F/\partial y_{i(calc)}^2)$ by differentiating the analytical expression for F twice.
- 15. We next calculate the n values of $(3^2F/3A3B)$, evaluated at $y_{i(calc)}$, which is obtained from the analytical expression for F.
- 16. We next calculate the n values of $(\partial^2 F/\partial A\partial C)$, evaluated at $y_{i(calc)}$, which is obtained from the analytical expression for F.
- 17. We next calculate the n values of $(\partial^2 F/\partial B\partial C)$, evaluated at $y_{i(calc)}$, which is obtained from the analytical expression for F.
- 18. We next calculate the n values of $(\partial^2 F/\partial y_i)$ (calc) ∂A which is obtained from the analytical expression for F.
- 19. We next calculate the n values of $(\partial^2 F/\partial y_{i(calc)}\partial B)$ which is obtained from the analytical expression for F.
- 20. We next calculate the n values of $(\partial^2 F/\partial y_{i(calc)}\partial C)$ which is obtained from the analytical expression for F.
- 21. We next calculate the n values of $(\partial Y_i/\partial A)$ using equation (49).
- 22. We next calculate the n values of $(\partial Y_i/\partial B)$ using equation (50).
- 23. We next calculate the n values of $(\partial Y_i/\partial C)$ using equation (51).

- 12. We next colculate the n values of (3 T/35), evaluated at Victoria; by differentiation of the analytical expression for P twice.
- II. We make calculate the n values of (2 1/20), evaluated at little of the respective of the restriction of

14. We news coloulage the n values of (6 T/6) 1(nale)) by

Lister relating one undergraded expression one (afficient water and it.).

The state of the north land of the state of

at Fiscaldy, obtains the contents the enginters, expression for I.

Valcate) which is secretaring from the analytical expression for F.

13. We note testentate the nivalues of (5 F/3) (cate) 50 which is

19. We make colouland the n values of (5 Froverence) and union is the obtained from the sequences of the Froverence of the Front from the sequence of the sequence of

20. We nest relociate the a values of (5 F13) ((cold)) which is obtained from the analytical appreciation for F.

21. We sawt relocated the a values of (SY, /SA) water aquetion

22. We maxt calculate the m values of (81,/83) using equation (50).

23. We next-colculate the a values of (67,/60) veing equation

- 24. We next calculate the n values of $(\partial^2 Y_i/\partial A^2)$ using equation (58).
- 25. We next calculate the n values of $(\partial^2 Y_i/\partial B^2)$ using equation (59).
- 26. We next calculate the n values of $(\partial^2 Y_i/\partial C^2)$ using equation (60).
- 27. We next calculate the n values of $(\partial^2 Y_i/\partial A\partial B)$ using equation (67).
- 28. We next calculate the n values of $(\frac{\partial^2 Y}{\partial A\partial C})$ using equation (68).
- 29. We next calculate the n values of $(\partial^2 Y_i/\partial B\partial C)$ using equation (69).
 - 30. We next calculate a from equation (19).
 - 31. We next calculate $a_2 = b_1$ from equation (20).
 - 32. We next calculate $a_3 = c_1$ from equation (21).
 - 33. We next calculate b₂ from equation (22).
 - 34. We next calculate $b_3 = c_2$ from equation (23).
 - 35. We next calculate c₃ from equation (24).
 - 36. We next calculate m, from equation (25).
 - 37. We next calculate m₂ from equation (26).
- 38. We next calculate m₃ from equation (27).
 - 39. We next calculate D₁ from equation (31).
- 40. We next calculate D₂ from equation (32).
 - 41. We next calculate D_3 from equation (33).
 - 42. We next calculate D_4 from equation (34).

- 24. We next culculate the n values of $(3^{2}V_{1}/3A^{2})$ nating equation
- 25. We desc delculate the n values of $(3^2Y_{\mu}/30^2)$ using equation
- 76. We next calculate the a value of (5"2, '50") using equation
- 27. We next culculate the n values of (8°4,/8A6B) using equation
- 25. We many reliculate the n values of (3'X' /3/3C) using equerion
- 29. We next calculate the a values of (3 x /3100) using equetion (69).
 - 30. We next cuttulate a, trop equation (19).
 - 31. We newt calculate s, s, from equation (20)
 - 12. We next calculate a, = c, from equation (21)
 - 39. We next calculate, b, from equation (22).
 - 34. We make calculate b c, from equation (23).
 - 35. We make calculate o, from equation (26)
 - 36. No news selection m, from equation (25)
 - 37. We nest calculate my from squarton (25)
 - 38. We ment calculate m, from equation (27)
 - 38. We make calculate D, from equation (31)
 - 40. We next colculate D, from equation (32)
 - 41. We next calculate D, from aquacton (33)
 - \$2. We next calculate D, from equation (34).

- 43. We next calculate D_5 from equation (35).
- 44. We next calculate D_6 from equation (36).
- 45. We next calculate D_7 from equation (37).
- 46. We next calculate D_8 from equation (38).
- 47. We next calculate Do from equation (39).
- 48. We next calculate D from equation (40).
- 49. We next calculate ΔA from equation (28).
- 50. We next calculate AB from equation (29).
- 51. We next calculate ΔC from equation (30).
- 52. We now return to step 1 and calculate $A_0 + \Delta A$, where A_0 is the value originally assumed in step 1.
- 53. We next calculate $B_0 + \Delta B$, where B_0 is the value originally assumed in step 1.
- 54. We next calculate $C_0 + \Delta C$, where C_0 is the value originally assumed in step 1.
- 55. Using $(A_0 + \Delta A)$, $(B_0 + \Delta B)$, and $(C_0 + \Delta C)$ as new values of A, B, and C, we proceed to step 2 and repeat steps 2 through 6.
- 56. At this point, we compare the value of R, the sum of the weighted squares of the residuals, with the initially calculated value of R. If it is smaller, we proceed to step 7 and repeat steps 7 through 55. We continue to repeat the iteration until $m_1 = m_2 = m_3 = 0$ within some predetermined small quantity. Our final values of A, B, and C are our least squares solution for these quantities.

- 43. We next calculate D, from equation (35).
 - AA. We next calculate De from squation (36)
 - 45. We mant colculate D. from equation (37)
- 46. We next calculate D, from equition (38)
- A7. We make calculate Do from equation (39)
- AB. We mast calculate D from equation (AO)
- AW. We must coloulate AA from equation (28).
- 50. We next celculate 38 from equation (29).
- 91. We next calculate AC from equation (30).
- 52. We now return to step 1 and calculate A + AA, where A is the value originally assumed in step 1.
- 35. We next calculate B, + AB, where B, is the value originally assumed in map 1.
- 56. We make colculate Co + MI, where Co is the value originally assumed to stop 1.
- 55. Dates (A, + AA), (B, + AB), and (C, + AC) as new values of A, B, and C, we proceed to step 2 and repeat steps 2 through b.

If in step 56 the sum of the weighted squares of the residuals, R, is larger than the initial value of R, the problem is diverging, and at this point, a technique must be used that will lead to convergence. One method that will lead to a smaller sum of R is the negative gradient or the method of steepest descent. The components of a vector in the direction of the negative gradient are given by

$$-\left(\frac{\partial R}{\partial A}\right)^{\circ}$$
; $-\left(\frac{\partial R}{\partial B}\right)^{\circ}$; and $-\left(\frac{\partial R}{\partial C}\right)^{\circ}$

Thus, if we take

$$\Delta A = -k \left(\frac{\partial R}{\partial A}\right)^{\circ} \tag{70}$$

$$\Delta B = -k \left(\frac{\partial R}{\partial B}\right)^{\circ} \tag{71}$$

$$\Delta C = -k \left(\frac{\partial R}{\partial C}\right)^{\circ} \tag{72}$$

where k is a positive constant, we will move in the direction of the negative gradient; k has been called the size of the step. We determine the size of the step in the following way. We expand R in a Taylor's series expansion, retaining terms through the second derivatives. Thus,

If in step 55 the sum of the weighted aquates of the residuals, a larger than the initial value of R, the problem is diverging.

and at this point, a rechnique must be used that will lead to comvergence. One method that will lead to a smaller sum of R is the negative gradient or the method of steepeast descent. The components of a vector in the direction of the negative gradient are given by

Thus, if we cake

$$(17)$$
 (86) -84

$$\Delta C = -4 \left(\frac{25}{36} \right) a = -24$$

where h is a positive constant, we will now in the direction of the negative gradient; h has been called the size of the atep. We determine the size of the second 2 in a laylor's suries expansion, retaining terms through the second derivatives. Thus,

$$R = \left(\frac{\partial R}{\partial A}\right)^{\circ} \Delta A + \left(\frac{\partial R}{\partial B}\right)^{\circ} \Delta B + \left(\frac{\partial R}{\partial C}\right)^{\circ} \Delta C$$

$$+ \left(\frac{\partial^{2} R}{\partial A^{2}}\right)^{\circ} (\Delta A)^{2} + \left(\frac{\partial^{2} R}{\partial B^{2}}\right)^{\circ} (\Delta B)^{2} + \left(\frac{\partial^{2} R}{\partial C^{2}}\right)^{\circ} (\Delta C)^{2}$$

$$+ \left(\frac{\partial^{2} R}{\partial A \partial B}\right)^{\circ} \Delta A \Delta B + \left(\frac{\partial^{2} R}{\partial A \partial C}\right)^{\circ} \Delta A \Delta C + \left(\frac{\partial^{2} R}{\partial B \partial C}\right)^{\circ} \Delta B \Delta C$$

$$(73)$$

We now substitute in equation (73) for $\triangle A$, $\triangle B$, and $\triangle C$ from equations (70), (71), and (72). The result is

$$R = + \frac{1}{2} k^{2} \left[\frac{\partial R}{\partial A} \right]^{0} + \left(\frac{\partial R}{\partial B} \right)^{0} + \left(\frac{\partial R}{\partial C} \right)^{0}$$

$$+ k^{2} \left[\left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial^{2} R}{\partial A \partial B} \right)^{0} + \left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial R}{\partial C} \right)^{0} \left(\frac{\partial^{2} R}{\partial A \partial C} \right)^{0} + \left(\frac{\partial R}{\partial B} \right)^{0} \left(\frac{\partial^{2} R}{\partial C} \right)^{0} \right]$$

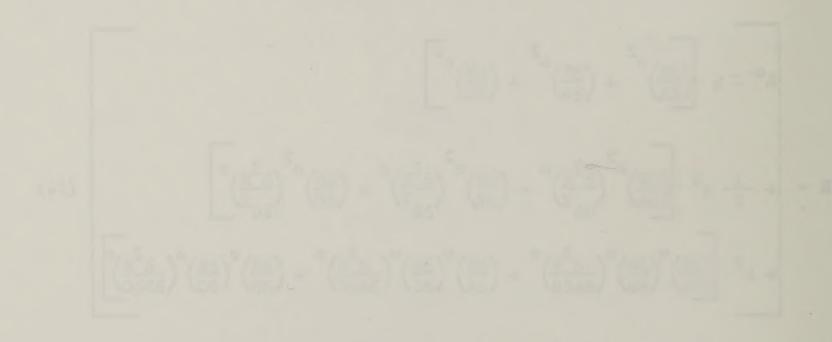
$$+ k^{2} \left[\left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial^{2} R}{\partial A \partial B} \right)^{0} + \left(\frac{\partial R}{\partial A} \right)^{0} \left(\frac{\partial^{2} R}{\partial A \partial C} \right)^{0} + \left(\frac{\partial R}{\partial B} \right)^{0} \left(\frac{\partial^{2} R}{\partial B \partial C} \right)^{0} \right]$$

$$(74)$$

We now differentiate equation (74) with regard to k, set the derivative equal to zero, and solve for k. The result is

$$k = \frac{\left(\frac{\partial R}{\partial A}\right)^{\circ 2} + \left(\frac{\partial R}{\partial B}\right)^{\circ 2} + \left(\frac{\partial R}{\partial C}\right)^{\circ 2}}{\left(\frac{\partial R}{\partial A}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial A}\right)^{\circ 2} + \left(\frac{\partial R}{\partial B}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + \left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2}} + 2\left(\frac{\partial R}{\partial A}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial A}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2} R}{\partial C}\right)^{\circ 2} + 2\left(\frac{\partial R}{\partial C}\right)^{\circ 2} \left(\frac{\partial^{2$$

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From equation (5) and equations (19) - (27), it is possible to show that

$$\left(\frac{\partial R}{\partial A}\right)^{O} = -2m_{1} \tag{76}$$

$$\left(\frac{\partial R}{\partial B}\right)^{O} = -2m_{2} \tag{77}$$

$$\left(\frac{\partial R}{\partial C}\right)^{O} = -2m_{3} \tag{78}$$

$$\left(\frac{\partial^2 R}{\partial A^2}\right)^{\circ} = 2a_1 \tag{79}$$

$$\left(\frac{\partial^2 R}{\partial B^2}\right) = 2b_2 \tag{80}$$

$$\left(\frac{\partial^2 R}{\partial C^2}\right)^{\circ} = 2c_3 \tag{81}$$

$$\left(\frac{\partial^2 R}{\partial A \partial B}\right)^0 = 2a_2 = 2b_1 \tag{82}$$

$$\left(\frac{\partial^2 R}{\partial A \partial C}\right)^{\circ} = 2a_3 = 2c_1 \tag{83}$$

$$\left(\frac{\partial^2 R}{\partial B \partial C}\right)^{\circ} = 2b_3 = 2c_2 \tag{84}$$

Substituting equations (76) - (84) into equation (75), we have

$$k = \frac{m_1^2 + m_2^2 + m_3^2}{2a_1m_1^2 + 2b_2m_2^2 + 2c_3m_3^2 + 4a_2m_1m_2 + 4a_3m_1m_3 + 4b_3m_2m_3}$$
(85)

From equation (5) and equations (19) - (27), it is possible

to show that

$$\left(\frac{2\pi}{2\lambda}\right)^{\alpha} = -2m_{\tilde{\chi}}$$
(76)

$$\left(\frac{36}{38}\right)^{2} = -2m_{2}$$

$$\left(\frac{\partial E}{\partial G}\right)^{\circ} = -2n_{\odot}$$
 (78)

$$\binom{3^{2}}{2}^{2} = 2a_{1}$$
 (29)

$$\binom{3.8}{38} = 282$$
 (80)

$$(2\frac{2^{2}g}{2c^{2}})^{2} = 2c_{3}$$
 (81)

$$\left(\frac{3^2 R}{3A3R}\right)^2 = 2a_2 = 2b_1$$
 (82)

$$\left(\frac{3^2 \pi}{3430}\right)^0 = 2\pi_3 = 2\sigma_1 = (83)$$

$$\left(\frac{3^28}{3880}\right)^0 = 2b_3 = 2c_2$$
 (84)

Substituting equations (76) - (84) into equation (75), we have

We see, therefore, that if the Newton-Raphson method is used in setting up the normal equations, the size of the step to be taken in the direction of the negative gradient can be evaluated very simply from the coefficients appearing in the normal equations, provided the calculation of k leads to a positive quantity. If equation (85) leads to a negative quantity, this means that the curvature of the surface is such that the trial solution is near a maximum and not a minimum. Under these conditions, the negative value of k must be ignored and positive values of k explored on a trial basis.

Of course, the same formal calculation can be made if the Gauss-Newton method is used to set up the normal equations. However, if one is in a region of divergence, this means that the trial solution is far from the true answer. Under these conditions, the residuals will be large and the second summation in the a's, b's, and c's, involving the second derivative terms, will be of importance compared to the first term in the summation. We therefore believe that if it is necessary to use the negative gradient method, it is better to use the Newton-Raphson method in setting up the normal equations and in calculating the size of the step.

If our problem is diverging after the first iteration, we calculate ΔA , ΔB , and ΔC from

$$\Delta A = 2km_1 \tag{86}$$

$$\Delta B = 2km_2 \tag{87}$$

$$\Delta C = 2km_3 \tag{88}$$

We see, therefore, that if the Newton-Raphson method is used in satting up the normal equations, the size of the step to be taken in the direction of the negative gradient can be evaluated very simply from the coefficients appearing in the normal equations, provided the calculation of k leads to a positive quantity. If equation (85) leads to a negative quantity, this means that the curvature of the surface is such that the trial solution is near a maximum and not a sinimum. Under these conditions, the negative value of k must be ignored and positive values of k explored on a trial basis.

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If our problem is diverging siler the liner iteration, we calculate the and to from

 $\Delta A = 2km_{\gamma}$ (86)

 $\Delta B = 2 km_2$ (87)

(88) (88)

with k calculated from equation (85). Using $A_{o} + \Delta A$, $B_{o} + \Delta B$, and $C_{o} + \Delta C$ as new values of our undetermined constants, A, B, and C, we then return to step 2 of the iteration. For each new value of A, B, and C, we solve the normal equations and calculate a new value of R. If we are diverging, we continue with the negative gradient method until the region of convergence is reached. As soon as this happens, we drop the negative gradient method and iterate by solving the normal equations for ΔA , ΔB , and ΔC . A scheme such as this should always lead to convergence.

Although the above scheme should always lead to convergence, the negative gradient method may be tediously slow in entering the region of convergence for the solution of the normal equations. Under these conditions, other schemes can be tried which may enter the region of convergence more rapidly than the negative gradient method.

Some of these methods are: (1) the Hartley (3) method; (2) a modification of the Hartley method due to Strand, Kohl, and Bonham (8); and (3) the method of Marquardt (4).

We do not have enough experience to judge the relative merits of these various methods. In our applications so far, we have not been troubled by lack of convergence.

CALCULATION OF VARIANCES AND COVARIANCES

With the value of our constants determined, the remaining questions to be answered are: (1) What are the variances and covariances of the constants evaluated? (2) What are the variances

view of the parties of our understand constants, A, B;

and C, we then various of our understands constants, A, B;

and C, we then various to simp 2 of the iteration. For each new

value of th, b, and C, we solve the moral equations and calculate

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We do not have emurgh experience to judge the relative merica of the converted as the our applications so the out bave not been troubled by lack of convergence.

CALCULATION OF VARIATIONS AND CONSTRUCTS

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of the calculated y 's and of any other calculated y that reduces F to zero?

To answer these questions, we apply the law for the propagation of errors. This law states that if we have a quantity or function, Q, that is a function of the independent quantities, y_1 , y_2 , y_3 , ..., then the variance of Q is given by

$$s_{\mathbf{Q}}^{2} = \sum_{i=1}^{n} \left(\frac{\partial Q}{\partial y_{i(0)}}\right)^{2} s_{y_{i(0)}}^{2}$$
(89)

where S_Q^2 is the variance of Q, and $S_{y_{i(0)}}^2$ is the variance of $y_{i(0)}$.

The value of the constant A that we have evaluated is a function of all of the observed x_i 's and of all of the observed y_i 's. Since we have assumed there are no random errors in the x_i 's, the variances of the x_i 's are zero. Then the expression for the variance of A is given by the equation

$$s_{A}^{2} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial y_{i(o)}}\right)^{2} s_{y_{i(o)}}^{2}$$
(90)

and there will be an equation similar to equation (90) for evaluating the variance of B and of C.

To evaluate equation (90), we must evaluate $(\partial A/\partial y_{i(0)})$ for each $y_{i(0)}$, multiply this quantity by $S_{y_{i(0)}}$, square the product, and then sum the product over all of the observed y_{i} 's.

Now we have a total of n pairs of the observed quantities \mathbf{x}_{i} , \mathbf{y}_{i} . Then our constants to be evaluated are determined by the

of the calculated y 's and of any other calculated y that reduces & to sero!

To unased these questions, we apply the law for the propagation of errors. Into low status that it we have a quantity or function, Q, that is a function of the independent quantities, y,,

where of is the variance of Q, and S, is the variance of Y1(0).

The value of the constant A that we have evaluated is a function of all of the observed Y,'s. Since we have essuad there are no random errors in the Wariances of the X,'s are sero. Then the expression for the variance of A is given by the equation

and there will be an equation similar to equation (90) for eval-

In evaluate equation (50), we must evaluate (50/5), for most fig.), for most fig.), multiply this quantity by S, , square the product, and then sum the product over all of the observed y, s.

Now we have a rotal of n pairs of the osserved quantities No.

solutions of equations (6), (7), and (8). Now suppose we change one of the y_i 's, $y_2(0)$ say, to $y_2(0) + \Delta y_2$. Then on solving equations (6), (7), and (8), we would get new values of (A + Δ A), (B + Δ B), and (C + Δ C) for our constants. Then we can calculate

$$\frac{\partial A}{\partial y_{2(0)}} = \frac{\Delta A}{\Delta y_{2}} \tag{91}$$

$$\frac{\partial B}{\partial y_{2(0)}} = \frac{\Delta B}{\Delta y_{2}} \tag{92}$$

$$\frac{\partial C}{\partial y_{2(0)}} = \frac{\Delta C}{\Delta y_{2}} \tag{93}$$

This means that when $y_{2(0)}$ is changed by a small amount, equations (6), (7), and (8) must still hold exactly. Mathematically, this means that

$$\frac{\partial}{\partial y_{i(o)}} \left(\frac{\partial R}{\partial A}\right)_{x_{i}, y_{i(o)}, B, C} = 0$$
 (94)

$$\frac{\partial}{\partial y_{i(o)}} \left(\frac{\partial R}{\partial B} \right)_{x_{i}, y_{i(o)}, A, C} = 0$$
 (95)

$$\frac{\partial}{\partial y_{i(o)}} \left(\frac{\partial R}{\partial C}\right)_{x_{i}, y_{i(o)}, A, B} = 0$$
 (96)

All of the Y_i's are explicit functions of the $y_{i(0)}$'s, the x_{i} 's, and the constants A, B, and C. In the application of equations (94),

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(95), and (96), the derivatives $(\partial R/\partial A)_{x_i,y_i(o)}$, B,C, $(\partial R/\partial B)_{x_i,y_i(o)},A,C$, and $(\partial R/\partial C)_{x_i,y_i(o)},A,B$ are to be considered functions of all of the y i(o)'s and of the constants evaluated, with A, B, and C being functions of all of the $y_{i(0)}$'s. Under these conditions,

$$\frac{\partial}{\partial y_{i(0)}} \left(\frac{\partial R}{\partial A}\right)_{x_{i}, y_{i(0)}, B, C} \\
x_{i}, y_{j \neq i}$$

$$\left(\frac{\partial^{2}R}{\partial A^{2}}\right)_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial A}{\partial y_{i}(o)}\right)_{y_{j}\neq i} + \left(\frac{\partial^{2}R}{\partial B\partial A}\right)_{x_{i},y_{i}(o)}, C \left(\frac{\partial B}{\partial y_{i}(o)}\right)_{y_{j}\neq i} = 0 \quad (97)$$

$$+ \left(\frac{\partial^{2}R}{\partial C\partial A}\right)_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial C}{\partial y_{i}(o)}\right)_{y_{j}\neq i} + \left(\frac{\partial^{2}R}{\partial A}\right)_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial C}{\partial y_{i}(o)}\right)_{x_{j}\neq i} + \left(\frac{\partial^{2}R}{\partial A}\right)_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial C}{\partial y_{i}(o)}\right)_{x_{i},y_{j}\neq i}, A, B, C$$

$$\frac{\partial}{\partial y_{i(0)}} \left(\frac{\partial R}{\partial B}\right)_{x_{i}, y_{i(0)}, A, C} =$$

$$\frac{\partial^{2}R}{\partial B^{2}}_{x_{i},y_{i}(o)},A,C^{\left(\frac{\partial B}{\partial y_{i}(o)}\right)}_{y_{j\neq i}} + \left(\frac{\partial^{2}R}{\partial A\partial B}\right)_{x_{i},y_{i}(o)},C^{\left(\frac{\partial A}{\partial y_{i}(o)}\right)}_{y_{j\neq i}} + \left(\frac{\partial^{2}R}{\partial C\partial B}\right)_{x_{i},y_{i}(o)},A^{\left(\frac{\partial C}{\partial y_{i}(o)}\right)}_{y_{j\neq i}} = 0 \quad (98)$$

(95), and (90), the derivatives (88/86), and (86/86) are to be considered (86/86). A. S. rol (91/2), A. C. rol (91/2), A. S. rol (91/2), and (98/86) are to be considered, with functions of the y₁(₀), and the y₁(₀),

$$\frac{\left(\frac{\partial^{2}R}{\partial c^{2}}\right)_{x_{i},y_{i(o)},A,B}\left(\frac{\partial c}{\partial y_{i(o)}}\right)_{y_{j}\neq i}}{+\left(\frac{\partial^{2}R}{\partial A\partial C}\right)_{x_{i},y_{i(o)},A}\left(\frac{\partial A}{\partial y_{i(o)}}\right)_{y_{j}\neq i}} = +\left(\frac{\partial^{2}R}{\partial B\partial C}\right)_{x_{i},y_{i(o)},A}\left(\frac{\partial B}{\partial y_{i(o)}}\right)_{y_{j}\neq i}} = 0 \quad (99)$$

$$+\left(\frac{\partial^{2}R}{\partial B\partial C}\right)_{x_{i},y_{i(o)},A}\left(\frac{\partial B}{\partial y_{i(o)}}\right)_{y_{j}\neq i}} = -0 \quad (99)$$

Equations (97), (98), and (99) contain terms involving the derivative of each constant with respect to $y_{i(0)}$ and can be solved by elementary algebra for the derivatives of the constants with respect to $y_{i(0)}$. These derivatives are then to be multiplied by $y_{i(0)}$, the product squared, and then the squared product is to be summed over all of the observed $y_{i(0)}$'s. These sums give us the variance of each constant evaluated.

Differentiating R, given by equation (5), by the constants A, B, and C keeping x_i and $y_{i(0)}$ constant, we have

$$\left(\frac{\partial R}{\partial A}\right)_{x_{i},y_{i}(o),B,C} = 2\sum_{i=1}^{n} w_{y_{i}(o)} Y_{i}\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i},y_{i}(o),B,C}$$
(100)

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Equations (37), (30), and (99) contain terms involving the derivative of each constant with respect to y₁(o) and can be solved by elementary elgebra for the derivatives of the constants with respect to y₁(o). These derivatives are then to be multiplied by summed ones all of the observed, and then the equated product is to be summed ones all of the observed y₁(o). These sums give, us the summed ones all of the observed y₁(o). These sums give, us the

D. amf C Keeping S, and y (6) to the constants A.

$$\left(\frac{\partial R}{\partial B}\right)_{x_{i},y_{i}(o),A,C} = 2\sum_{i=1}^{n} w_{y_{i}(o)} Y_{i}\left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{i},y_{i}(o),A,C}$$
(101)

$$\left(\frac{\partial R}{\partial C}\right)_{x_{i},y_{i}(o),A,B} = 2\sum_{i=1}^{n} w_{y_{i}(o)} Y_{i}\left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i},y_{i}(o),A,B}$$
(102)

Differentiating equation (100) with regard to each constant, we have

$$\left(\frac{\partial^{2}R}{\partial A^{2}}\right)_{x_{i},y_{i}(o)},B,C = 2\sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)},B,C + Y_{i}\left(\frac{\partial^{2}Y_{i}}{\partial A^{2}}\right)_{x_{i},y_{i}(o)},B,C\right]$$
(103)

$$\frac{\partial}{\partial B} \left(\frac{\partial R}{\partial A}\right)_{x_{i}, y_{i}(o)}, B, C$$

$$x_{i}, y_{i}(o), A, C$$

$$+ 2 \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{i}, y_{i}(o)}, A, C$$

$$+ 2 \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial}{\partial B}\right)_{x_{i}, y_{i}(o)}, B, C$$

$$+ 2 \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial}{\partial B}\right)_{x_{i}, y_{i}(o)}, B, C$$

$$x_{i}, y_{i}(o), A, C$$

$$x_{i}, y_{i}(o), A, C$$
(104)

$$\frac{\partial}{\partial C} \left(\frac{\partial R}{\partial A}\right)_{x_{i}, y_{i}(o)}, B, C \\
x_{i}, y_{i}(o), A, B =$$

$$= \begin{cases}
2 \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i}, y_{i}(o)}, A, B \\
+ 2 \sum_{i=1}^{n} w_{y_{i}(o)}, Y_{i} \left(\frac{\partial}{\partial C}\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i}, y_{i}(o)}, B, C \\
+ 2 \sum_{i=1}^{n} w_{y_{i}(o)}, Y_{i} \left(\frac{\partial}{\partial C}\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i}, y_{i}(o)}, B, C \\
\end{bmatrix}_{x_{i}, y_{i}(o)}, A, B$$
(105)

Differentiating equation (100) with regard to each constant, we have

Differentiating equation (100) with regard to a single $y_{i(0)}$, keeping A, B, and C constant, we have

$$\begin{bmatrix}
\frac{\partial (\partial R)}{\partial A} \\
\frac{\partial Y_{i}(0)}{\partial y_{i}(0)}
\end{bmatrix}_{x_{i},y_{j\neq i},A,B,C} = 2
\begin{bmatrix}
w_{y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial y_{i}(0)} \\
\frac{\partial Y_{i}}{\partial y_{i}(0)} \end{pmatrix}_{x_{i},y_{j\neq i},A,B,C} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\
\frac{\partial Y_{i}}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C$$

$$+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} Y_{i} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial A}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial Y_{i}}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial Y_{i}}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \\ \frac{\partial Y_{i}}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial y_{i}(0)} \begin{pmatrix} \frac{\partial}{\partial A} \\ \frac{\partial}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial A} \end{pmatrix}_{x_{i},y_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial A} \end{pmatrix}_{x_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial A} \end{pmatrix}_{x_{i}(0)},B,C} \\
+ w_{y_{i}(0)} \begin{bmatrix} \frac{\partial}{\partial A} \\ \frac{\partial}{\partial A} \end{pmatrix}$$

The right-hand side of equation (106) reduces to a single term, since a single $y_{i(0)}$ only appears in one term in the summation.

Now from equation (4)

$$Y_{i} = y_{i(0)} - y_{i(calc)}$$
 (4)

so that

$$\left(\frac{\partial Y_{i}}{\partial y_{i(0)}}\right)_{x_{i},y_{j\neq i},A,B,C} = 1$$
 (107)

and it follows that

$$\frac{\partial}{\partial A} \left(\frac{\partial Y_{i}}{\partial y_{i(o)}} \right)_{x_{i}, y_{j \neq i}, A, B, C} = 0$$
(108)

Substituting equations (107) and (108) into equation (106), we have

Differentialing equation (100) with ragard to a single yistor teating A, E, and C constant, we have

The right-hard side of equation (104) reduces to a single term, since a single Para, since a single Para, only appears in one term in the summetton.

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said it follows that

Substituting equations (107) and (188) into equation (106), we have

$$\begin{bmatrix}
\frac{\partial (\frac{\partial R}{\partial A})_{x_{i},y_{i(o)},B,C}}{\partial y_{i(o)}} \\
& = 2w_{y_{i(o)}} \\
\frac{\partial Y_{i}}{\partial A} \\
x_{i},y_{i(o)},B,C
\end{bmatrix}$$

$$= 2w_{y_{i(o)}} (\frac{\partial Y_{i}}{\partial A})_{x_{i},y_{i(o)},B,C}$$
(109)

Substituting equations (103), (104), (105), and (109) in equation (97), we get

$$\left(\frac{\partial A}{\partial y_{1}(o)}\right)_{y_{j\neq 1}} \sum_{i=1}^{n} w_{y_{1}(o)} \left[\left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{1},y_{1}(o)}, B, C + Y_{i}\left(\frac{\partial^{2}Y_{i}}{\partial A^{2}}\right)_{x_{1},y_{1}(o)}, B, C\right]$$

$$+ \left(\frac{\partial B}{\partial y_{1}(o)}\right)_{y_{j\neq i}} \left[\sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{1},y_{1}(o)}, A, C\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{1},y_{1}(o)}, B, C\right]$$

$$+ \left(\frac{\partial C}{\partial y_{1}(o)}\right)_{y_{j\neq i}} \left[\sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left[\frac{\partial}{\partial C} \left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{1},y_{1}(o)}, A, B\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{1},y_{1}(o)}, A, C\left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{1},y_{1}(o)}, A,$$

Subscitzucing managions (103), (104), (103), and (109) in equation

Upon differentiating equations (101) and (102) with regard to a single $y_{i(0)}$, we obtain two expressions similar to equation (110). We will write equation (110) and the two other equations as

$$a_{1}\left(\frac{\partial A}{\partial y_{i(0)}}\right)_{y_{j\neq i}} + b_{1}\left(\frac{\partial B}{\partial y_{i(0)}}\right)_{y_{j\neq i}} + c_{1}\left(\frac{\partial C}{\partial y_{i(0)}}\right)_{y_{j\neq i}} = n_{1} \quad (111)$$

$$a_{2}\left(\frac{\partial A}{\partial y_{i(o)}}\right)_{y_{j\neq i}} + b_{2}\left(\frac{\partial B}{\partial y_{i(o)}}\right)_{y_{j\neq i}} + c_{2}\left(\frac{\partial C}{\partial y_{i(o)}}\right)_{y_{j\neq i}} = n_{2} \quad (112)$$

$$a_{3}\left(\frac{\partial A}{\partial y_{i(0)}}\right)_{y_{j\neq i}} + b_{3}\left(\frac{\partial B}{\partial y_{i(0)}}\right)_{y_{j\neq i}} + c_{3}\left(\frac{\partial C}{\partial y_{i(0)}}\right)_{y_{j\neq i}} = n_{3} \quad (113)$$

where

$$a_{1} = \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A} \right)_{x_{i}, y_{i}(o), B, C}^{2} + Y_{i} \left(\frac{\partial^{2} Y_{i}}{\partial A^{2}} \right)_{x_{i}, y_{i}(o), B, C}$$
(114)

$$a_{2} = b_{1} = \begin{bmatrix} \sum_{i=1}^{n} w_{y_{i}(o)} \begin{pmatrix} \frac{\partial Y_{i}}{\partial B} \end{pmatrix}_{x_{i}, y_{i}(o)}, A, C \begin{pmatrix} \frac{\partial Y_{i}}{\partial A} \end{pmatrix}_{x_{i}, y_{i}(o)}, B, C \\ + \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \begin{pmatrix} \frac{\partial^{2} Y_{i}}{\partial B \partial A} \end{pmatrix}_{x_{i}, y_{i}(o)}, C$$

$$(115)$$

$$a_{3} = c_{1} = \begin{bmatrix} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i},y_{i}(o)}, A,B \left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i},y_{i}(o)}, B,C \\ + \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial^{2} Y_{i}}{\partial C \partial A}\right)_{x_{i},y_{i}(o)}, B \end{bmatrix}$$

$$(116)$$

Upon differentiating equations (101) and (102) with regard to a single Fi(o), we obtain two expressions similar to equation (110). We will write equation (110) and the two other equations as

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$$b_{2} = \sum_{i=1}^{n} w_{y_{i}(o)} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{2}_{x_{i}, y_{i}(o)}, A, C + Y_{i} \left(\frac{\partial^{2} Y_{i}}{\partial B^{2}} \right)_{x_{i}, y_{i}(o)}, A, C \right]$$
(117)

$$b_{3} = c_{2} = \begin{bmatrix} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{i},y_{i}(o)}, A, C \left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i},y_{i}(o)}, A, B \\ + \sum_{i=1}^{n} w_{y_{i}(o)} Y_{i} \left(\frac{\partial^{2} Y_{i}}{\partial C \partial B}\right)_{x_{i},y_{i}(o)}, A \end{bmatrix}$$

$$(118)$$

$$c_{3} = \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C} \right)^{2}_{X_{i}, Y_{i}(o), A, B} + Y_{i} \left(\frac{\partial^{2} Y_{i}}{\partial C^{2}} \right)_{X_{i}, Y_{i}(o), A, B}$$
(119)

$$n_{1} = -w_{y_{i(o)}} \left(\frac{\partial Y_{i}}{\partial A}\right)_{x_{i}, y_{i(o)}, B, C}$$
 (120)

$$n_2 = -w_{y_i(o)} \left(\frac{\partial Y_i}{\partial B}\right)_{x_i, y_{i(o)}, A, C}$$
 (121)

$$n_{3} = -w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i}, y_{i}(o), A, B}$$
(122)

Notice that the a's, b's, and c's defined by equations (114) - (119) are of the same form as the a's, b's, and c's defined by

(111) and the same form as the a's, b's, and c's defined by

equations (19) - (24) which appear in our normal equations. The difference between these two definitions is that equations (114) - (119) apply when the least squares solution is obtained while equations (19) - (24) apply to trial solutions. The a's, b's, and c's given by equations (114) - (119) can be considered as the values of the coefficients in the normal equations for the final iteration.

Solving equations (111), (112), and (113) simultaneously for $(\partial A/\partial y_{i(o)})$, $(\partial B/\partial y_{i(o)})$, and $(\partial C/\partial y_{i(o)})$, we get

$$D_{o}\left(\frac{\partial A}{\partial y_{i(o)}}\right) = D_{1}n_{1} + D_{2}n_{2} + D_{3}n_{3}$$
 (123)

$$D_{o}\left(\frac{\partial B}{\partial y_{i(o)}}\right) = D_{4}n_{1} + D_{5}n_{2} + D_{6}n_{3}$$
 (124)

$$D_{o}\left(\frac{\partial C}{\partial y_{i(o)}}\right) = D_{7}^{n_{1}} + D_{8}^{n_{2}} + D_{9}^{n_{3}}$$
 (125)

where

$$D_1 = b_2 c_3 - b_3 c_2 (126)$$

$$D_2 = b_3 c_1 - b_1 c_3 \tag{127}$$

$$D_3 = b_1 c_2 - b_2 c_1 \tag{128}$$

$$D_4 = a_3 c_2 - a_2 c_3 (129)$$

$$D_5 = a_1 c_3 - a_3 c_1 \tag{130}$$

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$$D_6 = a_2 c_1 - a_1 c_2 (131)$$

$$D_7 = a_2 b_3 - a_3 b_2 (132)$$

$$D_8 = a_3 b_1 - a_1 b_3 \tag{133}$$

$$D_{9} = a_{1}b_{2} - a_{2}b_{1} \tag{134}$$

$$D_{o} = a_{1}D_{1} + a_{2}D_{2} + a_{3}D_{3}$$
 (135)

Notice that the D's defined by equations (126) - (135) are of the same form as the D's defined by equations (31) - (40) and, for the last iteration, when the least squares solution is obtained, they will be identical.

Squaring equation (123) and multiplying by $s_{y_{i(0)}}^{2}$, we have

$$D_{0}^{2}\left(\frac{\partial A}{\partial y_{i}(0)}\right)^{2}s_{y_{i}(0)}^{2} = + 2D_{1}D_{2}n_{1}n_{2}s_{y_{i}(0)}^{2} + 2D_{1}D_{3}n_{1}n_{3}s_{y_{i}(0)}^{2} + 2D_{1}D_{3}n_{1}n_{3}s_{y_{i}(0)}^{2} + 2D_{2}D_{3}n_{2}n_{3}s_{y_{i}(0)}^{2} + 2D_{2}D_{3}n_{2}n_{3}s_{y_{i}(0)}^{2} + 2D_{2}D_{3}n_{2}n_{3}s_{y_{i}(0)}^{2}$$

$$+ 2D_{2}D_{3}n_{2}n_{3}s_{y_{i}(0)}^{2}$$

$$(136)$$

But from equation (120)

$$n_1^2 s_{y_{i(0)}}^2 = w_{y_{i(0)}}^2 \left(\frac{\partial Y_i}{\partial A}\right)_{x_i, y_{i(0)}, B, C}^2 s_{y_{i(0)}}^2$$
 (137)

$$D_{p} = a_{p}b_{p} - a_{q}b_{p}$$
 (132)

$$D_{g} = a_{1}b_{2} - a_{2}b_{1}$$
 (174)

$$D_0 = a_1D_1 + a_2D_2 + a_3D_3$$
 (135)

Motice that the D's defined by equations (126) - (135) are of the same form as the D's defined by equations (31) - (40) and, for the lest iteration, when the less squares solution is obtained, they will be identical.

Squaring equation (123) and multiplying by 5 , we have

$$\frac{\left(\frac{1}{2}\right)^{2}}{\left(\frac{1}{2}\right)^{2}} = \frac{1}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{$$

But from equation (120)

Now

$$s_{y_{i(0)}}^2 = \frac{L^2}{w_{y_{i(0)}}}$$
 (138)

where L is a constant. Substituting equation (138) in equation (137), we find that

$$n_1^2 S_{y_{i(0)}}^2 = L^2 w_{y_{i(0)}} \left(\frac{\partial Y_i}{\partial A}\right)^2_{x_i, y_{i(0)}, B, C}$$
 (139)

Similarly, we find that

$$n_2^2 s_{y_{i(0)}}^2 = L^2 w_{y_{i(0)}} \left(\frac{\partial Y_i}{\partial B}\right)^2_{x_i, y_{i(0)}, A, C}$$
 (140)

$$n_3^2 S_{y_{i(0)}}^2 = L^2 w_{y_{i(0)}} \left(\frac{\partial Y_i}{\partial C}\right)_{x_i, y_{i(0)}, A, B}^2$$
 (141)

$$n_1 n_2 S_{y_{i(0)}}^2 = L^2 w_{y_{i(0)}} \left(\frac{\partial Y_i}{\partial A}\right)_{x_i, y_{i(0)}, B, C} \left(\frac{\partial Y_i}{\partial B}\right)_{x_i, y_{i(0)}, A, C}$$
 (142)

$$n_1 n_3 s_{y_{i(0)}}^2 = L^2 w_{y_{i(0)}} \left(\frac{\partial Y_i}{\partial A}\right)_{x_i, y_{i(0)}, B, C} \left(\frac{\partial Y_i}{\partial C}\right)_{x_i, y_{i(0)}, A, B}$$
 (143)

$$n_{2}n_{3}s_{y_{i(o)}}^{2} = L^{2} w_{y_{i(o)}} \left(\frac{\partial Y_{i}}{\partial B}\right)_{x_{i},y_{i(o)},A,C} \left(\frac{\partial Y_{i}}{\partial C}\right)_{x_{i},y_{i(o)},A,B}$$
(144)

where E is a constant. Substituting equition (138) in equation (137), we find that

$$(219)$$
 $\frac{2}{1}$
 $\frac{2}{3}$
 $\frac{2}{3$

Similarly, we find that

$$\frac{28^{2}}{28^{2}} = \frac{1^{2} v_{11(0)}}{28^{2}} \left(\frac{37}{60}\right)_{X_{1}X_{1}(0)} \frac{28^{2}}{48^{2}}$$
(141)

Substituting equations (139), (140), (141), (142), (143), and (144) into equation (136) and then summing over all of the observed $y_{i(o)}$'s, we have

$$\frac{\left(\frac{\partial A}{\partial y_{i}(o)}\right)^{2} s_{y_{i}(o)}^{2}}{\left(\frac{\partial A}{\partial y_{i}(o)}\right)^{2} s_{y_{i}(o)}^{2}} = s_{A}^{2} = \frac{L^{2}}{D_{o}^{2}} + 2D_{1}D_{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2} s_{x_{i},y_{i}(o)}, A, C$$

$$+ 2D_{1}D_{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2} s_{x_{i},y_{i}(o)}, A, B$$

$$+ 2D_{1}D_{3} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) s_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial Y_{i}}{\partial B}\right) s_{x_{i},y_{i}(o)}, A, C$$

$$+ 2D_{2}D_{3} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) s_{x_{i},y_{i}(o)}, B, C \left(\frac{\partial Y_{i}}{\partial C}\right) s_{x_{i},y_{i}(o)}, A, B$$

$$+ 2D_{2}D_{3} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) s_{x_{i},y_{i}(o)}, A, C \left(\frac{\partial Y_{i}}{\partial C}\right) s_{x_{i},y_{i}(o)}, A, B$$

$$+ 2D_{2}D_{3} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right) s_{x_{i},y_{i}(o)}, A, C \left(\frac{\partial Y_{i}}{\partial C}\right) s_{x_{i},y_{i}(o)}, A, B$$

$$+ 2D_{2}D_{3} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right) s_{x_{i},y_{i}(o)}, A, C \left(\frac{\partial Y_{i}}{\partial C}\right) s_{x_{i},y_{i}(o)}, A, B$$

Supersturing equations (139), (140), (141), (142), (143), and (144) into equation (136) and then summing over all of the observed

Squaring equation (124), multiplying by S^2 and then summing, we find that

$$\sum_{i=1}^{n} \left(\frac{\partial B}{\partial y_{i(o)}}\right)^{2} s_{y_{i(o)}}^{2} = s_{B}^{2} = \frac{L^{2}}{p_{o}^{2}} + 2p_{d}^{2} p_{o}^{2} + 2p_{d}^{2} p_{o}^{2$$

Equaring equation (124), multiplying by S2 and then securing, in find that

2. A. (a) 1 × 1 × (26) 018 (a) 1 × 2 × 3 (a) (a) 1 × 3 (a) (a) 1 × 3 (a) 1 × 8, A. (6), Y. X. (60) 5, S. (6), Y. X. (6), Y. Z. d. as

Similarly, from equation (125), we find that

$$\frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial X_{i}}{\partial B}\right)^{2}_{x_{i},y_{i}(o)}, A, C} + \frac{D_{8}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2}_{x_{i},y_{i}(o)}, A, C}{\left(\frac{\partial X_{i}}{\partial B}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{9}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B}{\left(\frac{\partial X_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C} + \frac{D_{9}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, C} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, C} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, C} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, B, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, B} + \frac{D_{7}^{2} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i},y_{i}(o)}, A, C}{\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i},y_{i}(o)}, A, C}$$

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(TAI)

For the covariances, we find from equations (123), (124), and (125)

$$\begin{bmatrix} D_{1}D_{4} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2} \\ + D_{2}D_{5} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} \\ + D_{3}D_{6} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} \\ + D_{3}D_{6} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} \\ + D_{3}D_{6} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} \\ + D_{2}D_{4} + D_{1}D_{5} \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{4} + D_{1}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{1}D_{6} + D_{3}D_{4}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{5}) \sum_{i=1}^{n} w_{y_{i}(o)} \left(\frac{\partial Y_{i}}{\partial A}\right) \\ + (D_{2}D_{6} + D_{3}D_{6}) \\ + (D_{2}D_{6} + D_{3}D_{6})$$

For the covertances, we find from equations (123); (124), and

(125)

Sy = 52 = 52 = 6)24 (0) 145 (0) 246

$$\int_{1}^{n} \int_{1}^{n} \int_{1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i}, y_{i(0)}, B, C}$$

$$+ D_{2}D_{8} \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2}_{x_{i}, y_{i(0)}, A, C}$$

$$+ D_{3}D_{9} \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i}, y_{i(0)}, A, B}$$

$$+ (D_{1}D_{8} + D_{2}D_{7}) \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i}, y_{i(0)}, A, B}$$

$$+ (D_{1}D_{9} + D_{3}D_{7}) \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i}, y_{i(0)}, B, C} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i}, y_{i(0)}, A, C}$$

$$+ (D_{1}D_{9} + D_{3}D_{7}) \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial A}\right)^{2}_{x_{i}, y_{i(0)}, B, C} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i}, y_{i(0)}, A, C}$$

$$+ (D_{2}D_{9} + D_{3}D_{8}) \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2}_{x_{i}, y_{i(0)}, A, C} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i}, y_{i(0)}, A, C}$$

$$+ (D_{2}D_{9} + D_{3}D_{8}) \int_{i=1}^{n} \int_{1}^{w} y_{i(0)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2}_{x_{i}, y_{i(0)}, A, C} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2}_{x_{i}, y_{i(0)}, A, C}$$

 $\left(\frac{3A}{6y_1(a)}\right)\left(\frac{3C}{6y_1(a)}\right)^2y_1(a) = 8\frac{2}{AC} - \frac{L^2}{D^2}$

(691)

and finally,

$$\begin{array}{c} D_{4}D_{7}\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial A}\right)^{2}x_{i},y_{i}(o),B,C \\ \\ +D_{5}D_{8}\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial B}\right)^{2}x_{i},y_{i}(o),A,C \\ \\ +D_{6}D_{9}\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}x_{i},y_{i}(o),A,B \\ \\ +D_{6}D_{9}\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial C}\right)^{2}x_{i},y_{i}(o),A,B \\ \\ +(D_{4}D_{8}+D_{5}D_{7})\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial A}\right)x_{i},y_{i}(o),B,C\left(\frac{\partial Y_{i}}{\partial B}\right)x_{i},y_{i}(o),A,C \\ \\ +(D_{4}D_{9}+D_{6}D_{7})\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial A}\right)x_{i},y_{i}(o),B,C\left(\frac{\partial Y_{i}}{\partial C}\right)x_{i},y_{i}(o),A,C \\ \\ +(D_{5}D_{9}+D_{6}D_{8})\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial B}\right)x_{i},y_{i}(o),A,C \\ \\ +(D_{5}D_{9}+D_{6}D_{8})\sum_{i=1}^{n}w_{y_{i}(o)}\left(\frac{\partial Y_{i}}{\partial B}\right)x_{i},y_{i}(o),A,C \\ \\ \end{array}$$

(065)

Equations (145), (146), (147), (148), (149), and (150) allow us to calculate the variances and covariances of the constants evaluated.

Now $D_{_{\scriptsize O}}$ is the value of the determinate of the coefficients that appear in the final normal equations. In a linear problem, it is found that $D_{_{\scriptsize O}}$ may be factored from the right-hand side of equations (145) - (150) so that the expressions for the variances and covariances only involve $D_{_{\scriptsize O}}$ to the first power in the denominator. This is not true for a non-linear problem as can be shown from the exact derivation given above. The present method in use in calculating variances and covariances is therefore only an approximation. How good this approximation is can only be decided by testing it against the mathematically exact calculation given above. Since this is true, we prefer to calculate variances and covariances using the mathematically exact equations given above.

We now answer the question, how do we calculate the variance of a calculated y which reduces the function F to zero for a given x? The answer to this question is obtained as follows: y is a function of x, and through the constants evaluated, is a function of all of the observed $y_{i(o)}$'s and x_{i} 's. When we apply the law for the propagation of errors, we have

$$s_{y}^{2} = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial y_{i(0)}} \right)^{2} s_{y_{i(0)}}^{2}$$

$$y_{j \neq i}^{x, x, x}$$
(151)

Equations (165), (167), (168), (169), and (150) allow up to calculate the veriances and covariances of the constants evaluated.

New Do to the value of the determinate of the confidence that appear in the final vertal equations is a linear problem, it is found that Do may be factored from the right-hand side of equations (185) - (150) so that the expressions for the variances and covertances only involve Do to the first power in the denominator. This is not true for a non-linear problem as can be shown from the exact derivation given above. The present method in use in calculating variances and covariances is therefore only an approximation. How the method this approximation is can only be decided by testing it against the method active special calculation given above. Since this is unathermatically exact calculation given above. Since this is mathematically exact equations given above. Since this is

We now enswer the question, how do we calculate the variance of a calculated y which reduces the function F to sero for a given all The onewer to this question is obtained as follows: y is a function of x, and through the constants evaluated, is a function of all of the observed y₁₍₀₎'s and x₁'s. When we apply the law for the propagation of errors, we have

We calculate $\left(\frac{\partial y}{\partial y_i(o)}\right)_{j\neq i}$, from equation (1). Differentiating equation (1), we have

$$\frac{\left(\frac{\partial F}{\partial A}\right)_{x,y,B,C} \left(\frac{\partial A}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} + \left(\frac{\partial F}{\partial B}\right)_{x,y,A,C} \left(\frac{\partial B}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} = 0 \quad (152)$$

$$+ \left(\frac{\partial F}{\partial C}\right)_{x,y,A,B} \left(\frac{\partial C}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} + \left(\frac{\partial F}{\partial y}\right)_{x,A,B,C} \left(\frac{\partial V}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i},x_{i}} = 0 \quad (152)$$

Solving equation (152) for $\left(\frac{\partial y}{\partial y_{i(0)}}\right)_{y_{j\neq i},x_{i},x}$, we get

$$\left(\frac{\partial F}{\partial A}\right)_{x,y,B,C} \left(\frac{\partial A}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} = -\frac{1}{\left(\frac{\partial F}{\partial y}\right)_{x,A,B,C}} + \left(\frac{\partial F}{\partial B}\right)_{x,y,A,C} \left(\frac{\partial B}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} (153)$$

$$+ \left(\frac{\partial F}{\partial C}\right)_{x,y,A,B} \left(\frac{\partial C}{\partial y_{i(o)}}\right)_{y_{j\neq i},x_{i}} (153)$$

We calculate (av.) from squarion (1). Differentiaring equation (1), we have

Solving equation (152) for $\left(\frac{\sqrt{5}}{37}\right)$ and (521) coldanos galving x_{1}, x_{2}, x_{3}

$$(C21) = \frac{\left(\frac{A6}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right)}{2^{2} \times 141} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{(0)1} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{36} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{36} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5 \cdot 4} \times \left(\frac{36}{36} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36}\right) + \frac{1}{3 \cdot 5} \times \left(\frac{36}{36} \times 6\right)_{0, A, X} \left(\frac{36}{36} \times 6\right)_{$$

Squaring equation (153), multiplying by $s_{y_{i(0)}}^{2}$, and then summing over all the observed $y_{i(0)}$'s, we obtain

$$S_{y}^{2} = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial y_{i}(o)}\right) S_{y_{i}(o)}^{2} = \frac{1}{\left(\frac{\partial F}{\partial y}\right)_{x,A,B,C}^{2}}$$

$$s_{A}^{2} \left(\frac{\partial F}{\partial A}\right)_{x,y,B,C}^{2} + s_{B}^{2} \left(\frac{\partial F}{\partial B}\right)_{x,y,A,C}^{2}$$

$$+ s_{C}^{2} \left(\frac{\partial F}{\partial C}\right)_{x,y,A,B}^{2} + 2s_{AB}^{2} \left(\frac{\partial F}{\partial A}\right)_{x,y,B,C}^{2} \left(\frac{\partial F}{\partial B}\right)_{x,y,A,C}$$

$$+ 2s_{AC}^{2} \left(\frac{\partial F}{\partial A}\right)_{x,y,B,C}^{2} \left(\frac{\partial F}{\partial C}\right)_{x,y,A,B}$$

$$+ 2s_{BC}^{2} \left(\frac{\partial F}{\partial B}\right)_{x,y,A,C}^{2} \left(\frac{\partial F}{\partial C}\right)_{x,y,A,B}$$

(154)

where the variances and covariances in equation (154) are to be calculated from equations (145), (146), (147), (148), (149), and (150).

Equation (154) is the general formula for calculating the variance of a calculated y, regardless of the functional relationship between y and x and the parameters, A, B, and C.

This concludes the derivations of the formulas used by us in the solution of general non-linear least squares problems.

In a later paper, we will apply these general formulas to the reduction of PVT data obtained by the Burnett method.

Squaring equation (153), multiplying by S 1 , and then summing over all the observed y (0) s, we obtain

where the variances and coveriences in equation (154) are to be calculated from equations (165), (165), (167), (168), (169), and (150), Equation (154) is the percent formula for calculating the variance of a rejculated y, regardless of the functional relationship between

This conviction of general con-linear less: squares problems.

In a later paper, we will apply these general formulas to the reducation of EVT data obtained by the Burnett method.

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